Correspondence on the Comments for "Iterative Estimation of Sinusoidal Signal Parameters"

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***Abstract*—The method proposed in [1], referred to as iterative Quasi-Harmonic Model (iQHM), is an iterative frequency estimation technique. It is based on a linearized version of the frequency error between the true frequency and the initially provided frequency, while the estimation of its unknown parameters is performed by Least Squares (LS) method. In [2], a relationship between iQHM and Gauss-Newton (GN) method was presented. More specifically it was claimed that iQHM is actually equivalent to an approximate Gauss-Newton method (AGN). In this correspondence, we show that iQHM is actually equivalent to a sequential version of the GN method.**

***Index Terms*—Frequency estimation, iterative QHM, Gauss-Newton method**

I. A QUICK REVIEW OF THE GAUSS-NEWTON TYPE ALGORITHMS FOR FREQUENCY ESTIMATION

Following the notation used in [2] and considering the case of one-component signal as in [2], the complex mono-component sinusoidal model for a signal $x(t)$ is given by:

$$x(t) = ce^{j\omega t} \tag{1}$$

Considering as sampling frequency $1Hz$ and using N values of $x(t)$, the optimum values for c and ω are obtained by minimizing the modeling error:

$$\begin{aligned} \epsilon(c,\omega) &= \textstyle\sum_{t=0}^{N-1} |x(t) - ce^{j\omega t}|^2 \\ &= ||x - ca||^2 \end{aligned} \tag{2}$$

where $x = [x(0), \cdots, x(N-1)]^T$ and $a = [1\ \ e^{j\omega 1} \cdots e^{j\omega(N-1)}]^T$. Due to the nonlinear dependence of a on ω, minimizing $\epsilon(c,\omega)$ is a nonlinear optimization problem. Assuming an initial estimate for amplitude and frequency, (c^0, ω^0), the Gauss-Newton algorithm for the above minimization problem can be derived by linearizing $\epsilon(c,\omega)$ around that initial estimate. In [2], two linearizations are suggested which lead to two algorithms, namely the Gauss-Newton (GN) and the approximate GN (AGN) algorithms:

- Gauss-Newton (GN):

$$\epsilon(c,\omega) \approx x - (c^0a^0 + a^0(c - c^0) + c^0(\omega - \omega^0)d^0) \tag{3}$$

- Approximate Gauss-Newton (AGN):

$$\epsilon(c,\omega) \approx x - (ca^0 + c(\omega - \omega^0)d^0) \tag{4}$$

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where $d = [0\ je^{j\omega 1} \cdots (N-1)je^{j\omega(N-1)}]^T$ is the first order derivative of a with respect to ω, while a^0 and d^0 denote the values of a and d at $\omega = \omega^0$.
Minimizing the corresponding linearized error function with respect to ω, we get an update for the frequency estimate for each algorithm

- Gauss-Newton (GN):

$$\omega^1 = \omega^0 + \mathcal{R}\left\{\frac{x - \hat{c}a^0}{c^0d^0}\right\} \tag{5}$$

- Approximate Gauss-Newton (AGN):

$$\omega^1 = \omega^0 + \mathcal{R}\left\{\frac{x - \hat{c}a^0}{\hat{c}d^0}\right\} \tag{6}$$

where $\mathcal{R}$ denotes the real part of a complex number and where $\hat{c} = argmin||x - ca^0||^2$. Comparing (5) and (6), we see that there is a slight difference in the denominator of their second term; in (5), the previous value of c, c^0, is used, while in (6), the next estimate $\hat{c}$ of c is used. Therefore, there is a slight delay in updating (5) compared to (6). Otherwise, the two frequency updating formulas are basically the same.

Using the same notation, iQHM is given by [1]:

$$x(t) = (c + tb)e^{j\omega^0 t} \tag{7}$$

In iQHM, the term b is decomposed into two terms, ρ_1 and ρ_2, respectively parallel and perpendicular to c:

$$b = \rho_1 c + \rho_2 jc \tag{8}$$

Substituting (8) in (7) we get

$$x = (1 + t\rho_1)ca^0 + \rho_2 cd^0 \tag{9}$$

Under the assumption that $\rho_1 \approx 0$, then $x - ca^0 \approx \rho_2 cd^0$. Combining this result with (6), we get

$$\omega^1 = \omega^0 + \rho_2 \tag{10}$$

which is the same update mechanism for the frequency as this suggested in [1] (under however the assumption that $\rho_1 \approx 0$).

Next, using a slightly different notation and writing explicitly all the steps for the Gauss-Newton algorithm, we show that iQHM is equivalent to a "sequential" version of the GN method.

II. Equivalence between iQHM and GN for frequency estimation

Let's consider again a mono-component complex sinusoidal signal given by

$$x(t) = ce^{j\omega t}w(t), \qquad t = -N, ..., N \tag{11}$$

where c is the complex amplitude, ω is the angular frequency while $w(t)$ is a symmetric window function.

A. iQHM Method

In QHM, signal, $x(t)$, is modeled by

$$\tilde{x}(t) = (a + tb)e^{j\hat{\omega}t}w(t), \qquad t = -N, ..., N \tag{12}$$

where $\hat{\omega}$ is an initial estimate of the frequency, a is the complex amplitude and b is the complex slope. In iQHM, at the ith step ($i = 0, 1, ...$), complex amplitude, $a^{(i)}$, and complex slope, $b^{(i)}$, are computed through LS which provides the estimates

$$\begin{aligned} a^{(i)} &= \frac{\sum_{t=-N}^{N} w^2(t)x(t)e^x - j\omega^{(i)}t}{\sum_{t=-N}^{N} w^2(t)} \\ b^{(i)} &= \frac{\sum_{t=-N}^{N} tw^2(t)x(t)e^{-j\omega^{(i)}n}}{\sum_{t=-N}^{N} t^2 w^2(t)} \end{aligned} \tag{13}$$

with $\omega^{(0)} = \hat{\omega}$. Then, following [1], an estimate for (c, ω) is given by

$$\begin{aligned} c^{(i+1)} &= a^{(i)} \\ \omega^{(i+1)} &= \omega^{(i)} + \rho_2^{(i)} \end{aligned} \tag{14}$$

where ρ_2 was defined in (8) and is computed as

$$\begin{aligned} \rho_2^{(i)} &= \frac{\mathcal{R}\left\{a^{(i)}\right\}\mathcal{I}\left\{b^{(i)}\right\} - \mathcal{I}\left\{a^{(i)}\right\}\mathcal{R}\left\{b^{(i)}\right\}}{|a^{(i)}|^2} \\ &= \mathcal{R}\left\{\frac{-j\bar{a}^{(i)}b^{(i)}}{|a^{(i)}|^2}\right\} = \mathcal{R}\left\{\frac{-jb^{(i)}}{a^{(i)}}\right\} \end{aligned} \tag{15}$$

where $\mathcal{R}\{\cdot\}$ and $\mathcal{I}\{\cdot\}$ denote the real and imaginary parts of a complex number while $\bar{a}$ denote conjugation of a. Substituting (15) in (14) we get

$$\omega^{(i+1)} = \omega^{(i)} - \mathcal{R}\left\{\frac{jb^{(i)}}{a^{(i)}}\right\} \tag{16}$$

B. GN Method

Assuming also an initial estimate for the amplitude, $\hat{c} = c^{(0)}$, the GN method suggests the following updating algorithm

$$\begin{pmatrix} c^{(i+1)} \\ \omega^{(i+1)} \end{pmatrix} = \begin{pmatrix} c^{(i)} \\ \omega^{(i)} \end{pmatrix} + (J_i^H J_i)^{-1} J_i^H r_i \tag{17}$$

for $i = 0, 1, ...$, where J_i is a $(2N + 1) \times 2$ matrix given by

$$J_i = \begin{pmatrix} w(-N)e^{j\omega^{(i)}(-N)} & w(-N)c^{(i)}j(-N)e^{j\omega^{(i)}(-N)} \\ \vdots & \vdots \\ w(N)e^{j\omega^{(i)}N} & w(N)c^{(i)}jNe^{j\omega^{(i)}N} \end{pmatrix} \tag{18}$$

and r_i is a $(2N + 1) \times 1$ vector given by

$$r_i = \begin{pmatrix} w(-N)\left(x(-N) - c^{(i)}e^{j\omega^{(i}(-N)}\right) \\ \vdots \\ w(N)\left(x(N) - c^{(i)}e^{j\omega^{(i)}N}\right) \end{pmatrix} \tag{19}$$

Then, (17) equals to (20) which leads to

$$\begin{pmatrix} c^{(i+1)} \\ \omega^{(i+1)} \end{pmatrix} = \begin{pmatrix} \frac{\sum_{t=-N}^{N} w^2(t)x(t)e^{-j\omega^{(i)}t}}{\sum_{t=-N}^{N} w^2(t)} \\ \omega^{(i)} - \mathcal{R}\left\{\frac{j\sum_{t=-N}^{N} tw^2(t)x(t)e^{-j\omega^{(i)}t}}{c_1^{(i)}\sum_{t=-N}^{N} t^2w^2(t)}\right\} \end{pmatrix} \tag{21}$$

Please note that the real operator is applied on the frequency update equation because frequency is a real parameter. Furthermore, the update equation for the complex amplitude does not use the estimate of the previous step. This is expected since the LS method is linear for the complex amplitude [3], [4].

C. Relation Between the Two Methods

Using the parameters from iQHM, the GN iteration is given by

$$\begin{pmatrix} c^{(i+1)} \\ \omega^{(i+1)} \end{pmatrix} = \begin{pmatrix} a^{(i)} \\ \omega^{(i)} - \mathcal{R}\left\{\frac{jb^{(i)}}{a^{(i-1)}}\right\} \end{pmatrix} \tag{22}$$

which shows that there is a delay of one step for the estimation of the complex amplitude of the signal. However, if the estimation of the sinusoidal parameters in GN method is performed sequentially, i.e. firstly update the complex amplitude given the frequency of the previous step and then update the frequency given the updated complex amplitude, then, iQHM and GN method are equivalent. In order to compare both methods, we used as in [2] a data set consisting of a single complex sinusoid sampled at 1 Hz, with parameters $c = 1$ and $f = 0.2148$ Hz. Figure 1 and 2 depicts the RMSE of $\hat{f}$ and $\hat{a}$ respectively. It can be observed that both methods lead to similar RMSE both for the frequency estimation and for amplitude estimation. This result contrasts with the one obtained in [2], where the RMSE for amplitude estimation was found lower for GN than iQHM.

III. Conclusion

We showed that iQHM is equivalent to the sequential GN method for the mono-component case. This result can be generalized to the multi-component case even though it cannot be presented analytically as for the mono-component case.

References

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$$\begin{pmatrix} c^{(i+1)} \\ \omega^{(i+1)} \end{pmatrix} = \begin{pmatrix} c^{(i)} \\ \omega^{(i)} \end{pmatrix} + \begin{pmatrix} \sum_{t=-N}^{N} w^2(t) & 0 \\ 0 & |c^{(i)}|^2 \sum_{t=-N}^{N} t^2 w^2(t) \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=-N}^{N} w^2(t) x(t) e^{-j\omega^{(i)} t} - c^{(i)} \sum_{t=-N}^{N} w^2(t) \\ \mathcal{R}\left\{ -j\bar{c}^{(i)} \sum_{t=-N}^{N} t w^2(t) x(t) e^{-j\omega^{(i)} t} \right\} \end{pmatrix} \quad (20)$$

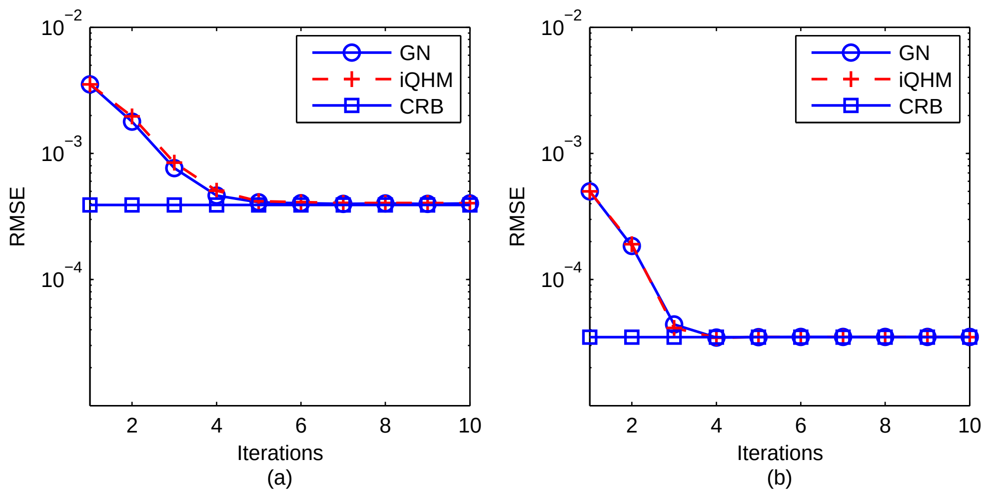

Fig. 1. RMSE of $\hat{f}$ vs. number of iterations for (a) $N = 100$, $SNR = 0$dB and (b) $N = 500$, $SNR = 0$dB. CRB denotes Cramer-Rao lower bound for frequency estimation.

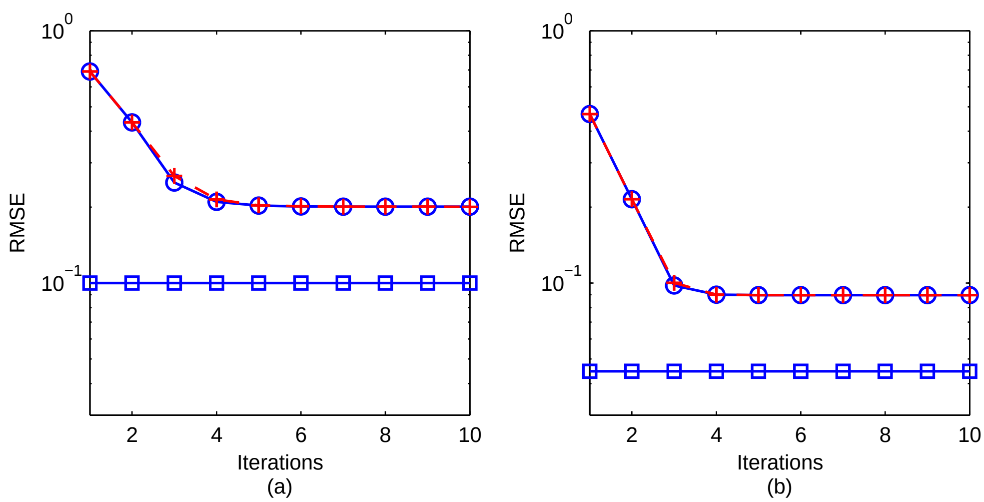

Fig. 2. RMSE of $\hat{c}$ vs. number of iterations for (a) $N = 100$, $SNR = 0$dB and (b) $N = 500$, $SNR = 0$dB. CRB denotes Cramer-Rao lower bound for amplitude estimation.