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IMAGE RECONSTRUCTION THROUGH THE PRIMATE CONE MOSAIC

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Abstract

We simulate the image reconstruction capacity of the actual foveal cone sampling mosaic of an adult monkey using classical Whittaker-Kotelnikov-Shannon sampling theory and a mapping of the cone lattice onto a perfectly hexagonal lattice. This simulation models the case where the positions of the photoreceptors are known. A two-dimensional hexagonal tessellation algorithm is described which permits a systematic study of lattice disorder and is particularly well suited for the cone mosaic. The reconstructed image quality achieved by our method is superior (as quantified by the mean-square error per pixel with respect to the input image) to that obtained by other methods which either do not compensate for the mosaic disorder or assume that the lattice is perfectly hexagonal. We find certain residual reconstruction artifacts that can be attributed to global mosaic irregularities. These artifacts have been minimized by applying our method locally to subregions within the mosaic which can be identified as small patches with a high degree of order bounded by points of mosaic disorder. The success of these local reconstructions raises the possibility of a physiological correlate based on local lattice order.
Introduction

The first step in visual information processing is image sampling by the retinal photoreceptors. Thus, the resulting image quality is influenced both by the sampling properties of the human retina and by the postreceptorial image reconstruction process. In this paper, the effects of the retinal photoreceptor topography on image quality are investigated on the basis of computer simulations. In the case of vision, the perceived image is assumed to be the result of the neural response to the sampled image intensities generated by the retinal photoreceptors. In our simulations, the neural response is modeled as a three-step process. First, the actual cone lattice is mapped onto a corresponding perfect hexagonal lattice. The mapping is one-to-one and preserves the adjacency of all lattice points. Second, an image is reconstructed by a Fourier transform operation using the ordered samples obtained at the previous step. Third, the mapping is reversed to obtain an image which corresponds to the image reconstructed through the actual cone mosaic.

The Retinal Mosaic

To study the quality of image reconstruction based on the primate photoreceptor mosaic we have taken the foveal cone mosaic from an adult monkey (Macaca fascicularis) using an electron micrograph of a section taken tangent to the external limiting membrane (ELM), near the presumed apertures of the photoreceptors. (See Miller [1] for histological details.) Our analysis of the image reconstruction capacity of this lattice is based on measurements of the positions of the centers of 388 cone inner segments within a retinal area of 56x56 μm and subtending 0.23 square degrees of foveal retina including the exact foveal center. Retinal eccentricity in degrees is based on a conversion factor of 243 μm/deg and assumes a focal length of 13.9 mm. Detailed analyses of this retinal window have been published by Hirsch and Miller [2]. The advantages of this lattice include: 1) it is sampled at the ELM which is the spatial aperture of the cone and its position specifies cone location for the purpose of image reconstruction (See Hirsch and Hylton [3]); 2) the analysis system is well developed for this lattice and we can take advantage of the previous calibrations and controls for this sample; and 3) since the fundamental anatomical features of this lattice are similar to other reported monkey lattices (Ferry and Cowey [4]; Schein [5]; de Monasterio et al [6]; and Borwein et al [7]) we can assume it is a sufficiently representative sample of primate mosaic to yield insight into the general problems of image reconstruction based on images sampled by the cone mosaic.

1Henceforth, we will refer to the physical appearance of the photoreceptors in an electron micrograph of an anatomical section as the cone mosaic and to the dot matrix representation of the cone centers as the cone lattice.
Retinal Mosaic Quality and Consequences for Image Reconstruction

Although there is a general consensus that the granularity of the photoreceptor lattice is a fundamental factor in image reconstruction, there are differing views regarding the consequences of sampling disorder. Analyses of actual disorder in the primate sampling mosaic for both a short range sample (Hirsch and Hylton [3]) and for a long range sample (Hirsch and Miller [2]) indicate that the mosaic packing is primarily triangular with varying granularity and position irregularity. For the actual retinal window employed in this study (window no. 1), the average spacing between cone centers is $0.0129 \pm 0.0014$ degrees (about 10% of the mean). Although the average angular separations between all nearest neighbors is 59.2 degrees of rotation (60 degrees is expected for a perfect lattice), the fractional angle disorder is about 13 percent (Hirsch and Miller [2]). Since the foveal region has the highest anatomical resolving power of the entire retina and is the region of retina actually employed for high resolution tasks, this window is the appropriate region of retina to be used in investigating the image reconstruction capability of the eye.

The effects of both sampling granularity and sampling irregularity depend entirely upon how the sampled output is registered by the central visual processing centers of the brain. Bossomaier et al [8] have identified the two extreme cases regarding lattice disorder: 1) random variations in cone positions are not represented in the subsequent neural processing and consequently image quality is compromised, and 2) positions of cones are known and therefore sampling disorder is compensated for by subsequent neural analyses. Image quality is not compromised in this case. We propose an algorithm which is equivalent to the "positions are known" case, and compare our reconstruction results with a "positions are unknown" reconstruction.

The Image Reconstruction Method

Basic Assumptions

The proposed image reconstruction method based on the cone mosaic does not represent a direct biological model of the early module of visual processing. Rather, it offers a quantitative approach to investigating the sampling properties of the cone mosaic and the various consequences for the quality of the reconstructed image.

The central foveal window used in this study is depicted in Fig. 1a. The test image is 'Lena', a 512X512x12 bit image commonly used in image processing research (shown in Fig. 1b).

The test image is initially lowpass-filtered (see Fig. 2) to eliminate the possibility of aliasing, thus isolating the effects of the reconstruction process on image quality. The low-

3
pass filter bandwidth has been selected to roughly match the conventional two-dimensional hexagonal Nyquist limit of the lattice. Averaged sampling of image intensities was obtained by simple averaging within non-overlapping circular disks centered at each cone position\(^2\), as shown in Fig. 3.

Due to the imperfect ordering of the cone lattice, one cannot apply the classic Whittaker-Kotelnikov-Shannon (WKS) reconstruction technique directly to the sampled data generated by retinal sampling. Several reconstruction methods, which deal with this problem, have been proposed [9,10,11,12]. Our image reconstruction method is based on ideas suggested by Papoulis [9] for one dimensional signals and Clark et al [10] for two-dimensional signals. The latter method is simple and more directly applicable to our problem. Soumekh [12] has also proposed a simple and accurate method which however is constrained by the requirement for an analytic description of the sampled data distribution. Chen [11] has derived a mathematically rigorous optimal solution which has the disadvantage of being computationally inefficient because it involves inverse matrix calculations of size equal to the square of the number of samples.

Specifically, to overcome the constraints imposed by the observed disorder in the sampling lattice, we employ a method which consists of the following three steps: 1) An appropriate mapping of spatial coordinates yields an ordered hexagonal lattice, 2) the WKS reconstruction technique is applied to the ordered hexagonal array of samples, and 3) the mapping of spatial coordinates is reversed to obtain the final reconstructed image from the image reconstructed at the previous step. These steps are discussed in more detail below.

**Ordering of the Lattice**

In what follows, the input image function \( f(\vec{x}) \) is assumed to be sampled at a finite number of lattice points given by the coordinate pairs:

\[
\vec{x}_n = (x_{n1}, x_{n2}), \quad n = 1, \ldots, N
\]

For the purpose of this study, these points correspond to the actual positions of retinal photoreceptors in the foveal and near foveal regions, where the lattice is disordered but nearly hexagonal. The input image used in this study is shown in Fig. 4, after sampling through the foveal cone retinal window.

To obtain the sampled image intensities on an ordered lattice, we define a continuous and one-to-one mapping \( \tau \), such that:

\(^2\)Point sampling and Gaussian weighting over the circular apertures was also tested. Averaged sampling was superior to point sampling, whereas Gaussian weighting was found to be equivalent to simple averaged sampling. For reasons of computational simplicity the later was selected.
\[ \vec{x}' = \tau(\vec{x}) : \vec{X} \to \vec{X}' \quad \vec{X}, \vec{X}' = \mathbb{R}^2 \]

and

\[ \vec{x}_n' = \tau(\vec{x}_n) \quad \text{and} \quad \vec{x}_n' = k_n \vec{a} + l_n \vec{b} \quad k_n, l_n \in \mathbb{Z} \quad (2) \]

For the particular application of sampling by the foveal and near-foveal lattice, the points \( \vec{x}_n' \) in \( \vec{X}' \) space are selected to constitute a perfect hexagonal lattice. Therefore, in Eq.(2), \( \vec{a} \) and \( \vec{b} \) are given by:

\[ \vec{a} = (1,0) \quad \text{and} \quad \vec{b} = (1/2, \sqrt{3}/2) \quad (3) \]

The resulting ordered lattice is shown in Fig. 5. Furthermore, the mapping of spatial coordinates can be reversed and the ordered samples \( g(\vec{x}_n') \) can be obtained from the known samples \( f(\vec{x}_n) \), i.e.

\[ g(\vec{x}_n') = f(\tau^{-1}(\vec{x}_n')) = f(\vec{x}_n) \quad (4) \]

To find a transformation that satisfies Eq.(2) is not a trivial task in the two-dimensional case. The method used to derive the coordinate mapping \( \tau \) is based on the theory developed in [10], but the tessellation algorithm described below was designed for our specific application and found to work well on the disordered cone lattices used in this study. The method can be classified as a generalized hexagonal mapping based on the Dirichlet tessellation. Specifically, the two-dimensional space is partitioned in non-overlapping triangles, whose vertices are the sampling points. Fig. 6 shows the tessellation of the foveal window used in this study. \( \tau \) maps each triangle in \( \vec{X} \) space to an equilateral triangle in \( \vec{X}' \) space (see Fig. 7). Under this mapping, adjacent triangles preserve their adjacency.

In the next section, a tessellation algorithm is described which is used to derive the spatial coordinate mapping \( \tau \) and to quantify the degree of disorder in foveal and near foveal cone lattices.

**Tessellation Algorithm**

The first step in applying the image reconstruction procedure described above is to find a tessellation which can be mapped to a perfect hexagonal lattice tessellation, while preserving the adjacency of all vertices. All regions of such a tessellation must have three
sides and its vertices must be junctions of six links [10]. Then, \( \tau \) maps every point in the disordered \( \tilde{X} \) space to a point in \( \tilde{X}' \) space, according to Eq. (2), as follows:

\[
\tilde{x}_n \rightarrow \tilde{x}_{k,l}^*,
\]

where \( \tilde{x}_n \in \tilde{X} \) and \( \tilde{x}_{k,l}^* = k\tilde{a} + l\tilde{b}, \tilde{x}_{k,l}^* \in \tilde{X}' \). For bookkeeping reasons and to simplify the notation, a point \( \tilde{x}_n \in \tilde{X} \) which is mapped to the point \( \tilde{x}_{k,l}^* \in \tilde{X}' \) will be denoted by \( \tilde{x}_{k,l} \).

Thus, each point \( \tilde{x}_{k,l} \) is labeled as the \( \tau \)-image of \( \tilde{x}_{k,l} \) and the correspondence between points in \( \tilde{X} \) and \( \tilde{X}' \) defines the mapping \( \tau \).

**Step 1**

To find such a tesselation, we first define a set of starting points in \( \tilde{X} \) such that their \( \tau \)-images are points on one of the three intrinsic axes of a perfect hexagonal lattice in \( \tilde{X}' \) space. Specifically, a pair of adjacent starting points are selected in \( \tilde{X} \) space and labeled \( \tilde{x}_{0,0} \) and \( \tilde{x}_{1,0} \). The remaining starting points, to the right of \( \tilde{x}_{1,0} \), are labeled \( \tilde{x}_{k,0} \), \( k = 2, \ldots, K_0 \), and are selected according to the following criteria:

1. \( \tilde{x}_{k,0} \) is one of the points \( \tilde{x}_n \) which have not already been assigned a \( \tau \)-image.
2. \( \tilde{x}_{k,0} \) is found within a minimum distance of radius \( r \) from \( \tilde{x}_{k-1,0} \).
3. Their incremental orientation \( \Delta \theta_k \) is minimum compared to other candidates that meet criteria 1 and 2, where

\[
\Delta \theta_k = | \theta_k - \frac{\sum_{i=1}^{k-1} \theta_i}{k-1} |
\]

and

\[
\theta_k = \text{ang}(\tilde{x}_{k-1,0}, \tilde{x}_{k,0})
\]

is the angle of the line segment \( \tilde{x}_{k-1,0}, \tilde{x}_{k,0} \) relative to a standard reference orientation.

This process is repeated for points \( \tilde{x}_{k,0} \) with \( k = -1, \ldots, -K_0' \) located to the left of the points \( \tilde{x}_{k+1,0} \).

When this step of the algorithm is completed, the points \( \tilde{x}_{k,0} \) will have been selected such that \( \tau(\tilde{x}_{k,0}) = k\tilde{a}, \quad k = -K_0', \ldots, K_0 \) (see Fig. 12), where \( K_0' + K_0 + 1 \) is the total number of points on the first line.

**Step 2**
The second step of this algorithm involves the definition of the next set of points in $\tilde{X}$ such that their $\tau$-images constitute lines which are parallel to the first line in $\tilde{X}'$, as selected in the previous step of the algorithm. Specifically, for every two points $\tilde{x}_{k,0}$, $\tilde{x}_{k+1,0}$ $k = -K_0, \ldots, K_0$,
select the $\tilde{x}_{k,1}$ point, belonging to the next line in $\tilde{X}'$, such that:

1. It is one of the points $\tilde{x}_n$ which have not already been assigned a $\tau$-image.
2. The sum $|\tilde{x}_{k,0} - \tilde{x}_{k,1}| + |\tilde{x}_{k+1,0} - \tilde{x}_{k,1}|$
is minimum.

This procedure is repeated until all points $\tilde{x}_n$ have been assigned $\tau$-images $\tilde{x}'_{k,l}$, $k = -K_1', \ldots, K_1'$ and $l = -L', \ldots, L'$, where $L' + L + 1$ is the total number of lines. In the general case, the sum in criterion 2 is written in the form:

$$|\tilde{x}_{k,l} - \tilde{x}_{k,l+1}| + |\tilde{x}_{k+1,l} - \tilde{x}_{k,l+1}|$$

An illustration of the definition of a line parallel to the one defined in step 1 is shown in Fig. 13.

When step 2 has been completed, the mapping $\tau$ is readily available by construction. A triplet $\{\tilde{x}_{k,0}, \tilde{x}_{k+1,0}, \tilde{x}_{k+1,1}\}$ can now be mapped to the vertices of an equilateral triangle as can be seen in Fig. 7. Specifically, each point $\tilde{x}_{k,l} \in \tilde{X}$ is mapped to $\tau(\tilde{x}_{k,l}) = k\bar{c} + l\bar{b}$.

Image Reconstruction:
Ordered Hexagonal Lattice

The result of applying the classical WKS reconstruction technique to the ordered samples in $\tilde{X}'$ space, as generated by the algorithm described in the previous section, is shown in Fig. 8. In this space, $g(x'_n)$ are the sampled image intensities mapped onto a perfect hexagonal lattice and:

$$g(x') = \sum_{n=1}^{N} g(x'_n) \phi(x' - x'_n),$$

where the kernel $\phi(x')$ represents a lowpass filter and its shape is dependent on the sampling lattice. In the frequency domain $\tilde{U}'$, the corresponding lowpass filter $\Phi(\tilde{w}')$ is defined as:
\[ \Phi(\vec{u'}) = \begin{cases} 1 & \text{if } \vec{u'} \in \Omega \\ 0 & \text{otherwise} \end{cases} \]  

where \( \Omega \) is the hexagon defined by the vectors

\[ \vec{e}_1 = (2/3, 0), \quad \vec{d}_1 = (1/3, \sqrt{3}/3), \]  

i.e. the hexagon defined by the points \( \vec{e}_1, -\vec{e}_1, \vec{d}_1, -\vec{d}_1, \vec{e}_1, \) and \( \vec{d}_1 + \vec{e}_1 \) in Fig. 9.

This results from the fact that, in the spatial frequency domain, the reciprocal lattice of a hexagonal lattice, defined by Eqs. (2) and (3), is also hexagonal and is generated by the vectors

\[ \vec{e} = (0, 2\sqrt{3}/3), \quad \vec{d} = (1, \sqrt{3}/3). \]

The interpolation operation represented by Eq. (5) has been implemented in the spatial frequency domain for reasons of computational efficiency.

That the reconstructed image in Fig. 8 has the correct form can be confirmed by comparing it with the image shown in Fig. 10, which has been obtained by direct application of the mapping \( \tau \) to the input image in Fig. 2.

Image Reconstruction:
Actual Cone Lattice with Cone Positions Known

Once an image has been reconstructed in the \( \vec{X'} \) domain, the original coordinate mapping \( \tau \) can be used to obtain the final reconstructed image by mapping the computed image intensities back onto the actual cone lattice. Specifically,

\[ f(\vec{x}) = g(\tau(\vec{x})) \]  

or, using Eq. (5),

\[ f(\vec{x}) = \sum_{n=1}^{N} f(\vec{z}_n) \phi(\tau(\vec{x}) - \tau(\vec{z}_n)). \]  

However, the triangular tessellation defines the mapping function \( \tau \) only at the sampling points \( \vec{z}_n \). In order to obtain \( \tau \) for every point \( \vec{x} \in \vec{X} \), one must interpolate between the \( \tau(\vec{z}_n) \) values. This has been achieved using trilinear interpolation because of its simplicity and other reasons explained in [10]. The resulting image is shown in Fig. 11.
Reconstruction Artifacts

Image reconstruction in the $\mathbf{X}$ and, consequently, $\mathbf{X}$ space is not error free. Potential sources of error are:

1. Accumulated roundoff errors due to the many stages of computation involved in the implementation of the tessellation algorithm and the WKS reconstruction technique.

2. Aliasing. It can be shown that, if a function $f(\mathbf{z}), \mathbf{z} \in \mathbf{X}$ has the Fourier transform $F(\mathbf{u})$, then its Fourier transform in $\mathbf{X}$ will be:

$$G(\mathbf{u'}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{j2\pi (\mathbf{u} \cdot \tau^{-1}(\mathbf{z}))} F(\mathbf{u}) \ d\mathbf{u} \ e^{-j2\pi (\mathbf{u'} \cdot \mathbf{z'})} \ d\mathbf{z'} \quad (10)$$

or

$$G(\mathbf{u'}) = \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}^2} e^{j2\pi (\mathbf{u} \cdot \tau^{-1}(\mathbf{z}))} \ e^{-j2\pi (\mathbf{u'} \cdot \mathbf{z'})} \ d\mathbf{z'} \right\} F(\mathbf{u}) \ d\mathbf{u} \quad (11)$$

Therefore, an image function which is bandlimited in $\mathbf{X}$ is not necessarily bandlimited in $\mathbf{X}$. The opposite may also occur, but this is desirable in that it would lead to aliasing-free reconstruction. For a given sampling lattice, which of the two cases will occur is entirely dependent on the mapping $\tau$. This is due to the $\tau$ dependence of the expression in brackets in Eq. (11).

Results

Global Image Reconstruction

Although the global image reconstruction technique described in this report does not necessarily have a physiological correlate, it can be used to evaluate the validity of assumptions which are often made regarding the effects of sampling disorder on reconstructed image quality. This is particularly true for the foveal and near foveal cone lattices. This approach to image reconstruction assumes that cone positions are deterministically known to the higher levels of visual processing and, henceforth, will be referred to as CPK (Cone Positions Known) reconstruction.

A second reconstruction method, which has been implemented and compared with the CPK reconstruction method, is based on the assumption that the structural disorder of the cone lattice is simply an anti-aliasing mechanism and, therefore, the actual cone positions are not directly involved in the image reconstruction process, but are significant
only in terms of their global frequency domain characteristics [13]. We include this method as an example of how cone lattice disorder can play an indirect role in image reconstruction without the actual cone positions being taken into account explicitly. We refer to this case as the “cone positions effectively ignored (CPI)” reconstruction. In this case, image reconstruction is implemented as a simple lowpass operation on the Fourier transform of the disordered cone lattice, as shown in Fig. 14. The resulting image is shown in Fig. 15.

As a control, the WKS reconstruction technique was also applied to samples of the input image on a simulated perfect hexagonal lattice, with spacing equal to the average spacing of the foveal cones in the particular retina used in this study. The resulting reconstructed image is shown in Fig. 16 and, henceforth, it will be referred to as the “control” reconstruction.

The mean-square errors per pixel, relative to the lowpass-filtered input image, are summarized in Table 1 for the “control”, CPI, and CPK reconstruction methods under both aliasing and aliasing-free conditions. Aliasing was introduced by allowing the maximum spatial frequency of the input image to exceed the conventional Nyquist limit of the two-dimensional hexagonal lattice. Furthermore, the effects of point and averaged sampling on reconstructed image quality were considered and compared. In the case of averaged sampling, the sampling apertures used are the nonoverlapping circular disks shown in Fig. 3. It is observed that the anti-aliasing role of averaged sampling is not significant, in agreement with Williams [14].

The CPK reconstruction method compares rather favorably with the “control” method and performs much better than the CPI method. This can also be seen by inspection of Figs. 10, 15, and 16. The poor performance of the CPI reconstruction method is in agreement with views expressed by French, Snyder, and Stevenga [15].

In Fig. 17, the triangular tessellation of the disordered lattice has been superimposed on the image reconstructed by the CPK method to demonstrate that reconstructed image quality is critically dependent on the tessellation algorithm. As already mentioned above, image reconstruction in the $X'$ domain is not error-free and, therefore, the resulting reconstructions in $X$ space cannot be free of errors either. The actual reconstruction arti-

<table>
<thead>
<tr>
<th></th>
<th>Point sampling</th>
<th>Averaging</th>
<th>Point sampling</th>
<th>Averaging</th>
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</tr>
<tr>
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<td>25.4%</td>
<td>28.9%</td>
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</tr>
<tr>
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<td>7.5%</td>
<td>8.0%</td>
<td>15.5%</td>
<td>14.7%</td>
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Table 1: Reconstruction errors
facts are closely related to the spatial mapping \( \tau \), which in turn is related to the degree of disorder in the cone lattice.

Finally, under the assumption that cone lattice disorder is ignored by the higher levels of visual processing and the sampled image intensities are arbitrarily mapped onto a perfect hexagonal lattice, one obtains the result shown in Fig. 8, which is the image corresponding to \( g(x^\prime) \) in Eq.(5). The observed distortion of the reconstructed image is in agreement with claims in [15]. An important implication of this result would appear to be that cone positions are not ignored, but are encoded and used by the higher levels of visual processing in image reconstruction. The particular encoding scheme used and whether it is deterministic or stochastic in nature is a separate problem which ought to be investigated.

Local Image Reconstruction

The tessellation algorithm presented above is a powerful tool, rather than a method guaranteed to give a unique tessellation for each window of photoreceptors. In fact, the algorithm would yield two different tessellations for two different pairs of starting points. However, the resulting triangular tessellation can be used to quantify the degree of local disorder in the cone lattice based on the quantitative similarity of each of its hexagonal components to a perfect hexagon and measures of local or average spacing.

In addition to constructing a tessellation for a whole window of the cone lattice, it is possible to identify subregions within such windows for which a unique tessellation can be computed, independent of tessellations in other subregions. These subregions exhibit an almost perfect hexagonal order and cannot be combined with other subregions without being distorted. This suggests a possibly new functional approach to modeling the topography of retinal photoreceptors based on local tessellations with unique characteristics. An example of applying this tessellation approach to the first two foveal windows (macaca fascicularis) is shown in Fig. 18.

The above observations on the structure of the foveal cone lattice have led to the idea of a local WKS reconstruction. The results of two such subregions identified in the foveal window (see Fig. 19) are presented in Fig. 20. The tessellation algorithm assigns six neighbors to each lattice point \( \mathbf{x}_i \), with relative orientations \( \theta_{11}, \theta_{12}, \theta_{13}, \theta_{14}, \theta_{15}, \) and \( \theta_{16} \) respectively. The mean relative orientation is defined over all the lattice points as follows:

\[
\overline{\theta} = \frac{1}{6N} \sum_{i=1}^{N} \sum_{j=1}^{6} (\theta_{ij} - \theta_{ref})
\]

where \( \theta_{ref} = 0, 60, 120, 180, 240, \) and 300 degrees respectively. Then, the variance of the relative orientations is obtained as:
\[ \sigma_{\theta_r}^2 = \text{var}(\theta_r) = \frac{1}{6N} \sum_{i=1}^{N} \sum_{j=1}^{S} (\theta_{ij} - \theta_{\text{ref}j} - \bar{\theta}_r)^2, \]

For these subregions, a quantitative analysis of the lattice gives mean relative orientations of 17 and 34 degrees and corresponding standard deviations (\(\sigma_{\theta_r}\)) of 4.9 and 5.5 degrees respectively, as compared to a standard deviation of 28 degrees for the whole first window. Thus, the two subregions are almost regular and their orientations differ by 17 degrees. The corresponding root mean square reconstruction error is 5.5% for point sampling and 5.8% for averaged sampling with no aliasing. This error is the smallest one achieved among the methods which have been evaluated and is almost equal to the error for the "control" reconstruction.

**Discussion**

Using the functional tessellation approach introduced in the previous section, it may be possible to define a functional tessellation for the fovea and the whole retina, assuming that more retinal strips are available. An example of such a functional windowing scheme is shown in Fig. 21. In this figure, one can identify several subregions, each with a well-defined average spacing and orientation. Orientation differences between such subregions can also be observed. In particular, an ordered closed subregion which is approximately elliptical in shape can be seen between windows 1 and 2 of the Hirsch and Miller [2] lattice. These subregions exhibit hexagonal order and, therefore, image reconstruction can be achieved without significant errors. They are the candidate sites for local WKS reconstruction, as was implemented in section 5.2 (Figs. 19 and 20).

Local reconstructions within each functional window, with strict adherence to the sampling rules, could possibly provide answers to the intriguing question of human visual performance surpassing the locally implied 2-D hexagonal Nyquist limits (hyperacuity/acytopy issue). New limits could be calculated on the basis of subregions such as the ones shown in Fig. 21. The orientation shifts among the subregions, with respect to the three major hexagonal axes, could account for such phenomena.

An important outcome of the present study is the "localized" nature of hexagonal order. If cone positioning data in retinal strips adjacent to the central one is used, it may be possible to characterize the retinal sampling process on the basis of sampling subfields and to find a physiological correlate.

CPK reconstruction in these subregions requires further investigation. The proposed local image reconstruction method is also in agreement with Gabor-like analysis, as recently
proposed with respect to spatial vision tasks [16,17,18,19,20]. According to this type of analysis, cells in the primary visual cortex are assumed to represent the Gabor coefficients of the input image. An attempt to define and characterize a new spatial sampling scheme for the retina could be motivated by a possible analogy to its spatial frequency counterpart in the primary cortex. Gabor analysis, as a model of spatial vision, is the mathematical formalism corresponding to the experimental observation that frequency and size selective mechanisms are local rather than global in nature [18]. This is also compatible with the notion that the accuracy of the human hyperacuity/acyuity performance is a function of the prevailing spatial frequency of the stimulus [21,22,23].

The Gabor representation scheme, seen as a spatial vision model, possesses certain elegant features. Theoretically, it represents a unification of the global vs. local spatial processing theories [24]. The reduction in spatial redundancy, i.e. compression, it offers is substantial [25,26,27] and it matches the "natural" image statistics [28]. Channel sizes and the spacing of sensors within the channels are optimized by the spatial frequency and space uncertainty principle [25,29]. The regularity of cone positioning in the subregions serves two-dimensional localization and metric relationships [21,22], while the spectral analysis (i.e. texture) is facilitated [25].

Conclusion

This research can be extended towards an investigation of the functional retinal windowing scheme and, more generally, the role of local processing in image reconstruction and spatial discrimination tasks. Thus far, theorists have only considered simplified retinal models and experimental vision research has not employed quantitative image representation and reconstruction techniques. In pursuing this work, we hope to derive a functional model of the human visual system so that we can understand its remarkable performance in standardized experiments. This would also increase our understanding of the issues involved in parallel implementations of intermediate and high level image analysis tasks and may in turn lead to improvements in the designs of computer vision systems.
References


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Figure Captions

Figure 1: a) Foveal cone mosaic (window no.1 from [2]). b) Test image.

Figure 2: The lowpass-filtered input image.

Figure 3: Sampling by averaging over circular disks.

Figure 4: Disordered samples of the input image. The samples correspond to the lattice points of Figure 1.

Figure 5: Disordered samples of the input image mapped onto an ordered hexagonal lattice.

Figure 6: Two-dimensional hexagonal tessellation of the foveal cone lattice.

Figure 7: The $r$-transform maps each triangle defined by the tessellation to an equilateral triangle. The interior points of such a triangle are assigned to the interior points of the corresponding equilateral triangle using a trilinear interpolation function.

Figure 8: WKS reconstruction in the ordered hexagonal space, with appropriate interpolation between the ordered samples defined by Eq.(5).

Figure 9: The exterior hexagon is defined by the vectors $\mathbf{e} = (0, 2\sqrt{3}/3)$, $\mathbf{a} = (1, \sqrt{3}/3)$. These vectors generate the lattice that is the Fourier transform of the lattice generated by vectors $\mathbf{a}$ and $\mathbf{b}$. The interior hexagon is the lowpass kernel. If we sample a function by the lattice generated by $\mathbf{a}$ and $\mathbf{b}$, then aliasing will not occur if its spectrum belongs within the area of the lowpass kernel.

Figure 10: This image has been generated by direct application of the spatial mapping $r$ on the input image. It serves as a control for the WKS reconstruction in the ordered hexagonal space (Figure 8).

Figure 11: The final reconstructed image according to the proposed method. The samples correspond to the lattice points of Figure 4.

Figure 12: Step 1 of tessellation algorithm: Selection of the first set of points that will be assigned to a line in $X'$ space.

Figure 13: Step 2 of tessellation algorithm: Selection of the next set of points that will be assigned to the next line in $X'$ space. Each point is selected in such a way as to preserve the two-dimensional hexagonal topography.

Figure 14: The discrete Fourier transform of image samples on the disordered cone lattice (Figure 4). A CPI image reconstruction can be obtained by circular lowpass filtering in the frequency domain, followed by an inverse Fourier transform.
Figure 15: Result of CPI image reconstruction.

Figure 16: Result of "control" reconstruction.

Figure 17: The tessellated foveal cone lattice superimposed on the image reconstructed by the CPK method. This picture shows the dependence of reconstructed image quality on the spatial mapping $r$.

Figure 18: "Functional" tessellation of the first two foveal windows. Squares indicate 'seeds' of disorder and circumscribe ordered subregions.

Figure 19: The two foveal subregions, defined according to a "functional" approach to tessellation, superimposed on the input image.

Figure 20: a) Result of "local" image reconstruction for the two regions appearing in Figure 19. b) Input image with regions in (a) shown for comparison.

Figure 21: A possible new "functional" windowing scheme for the first three foveal windows. Windows are defined with respect to two-dimensional hexagonal uniformity.
Figure 1. a) Foveal cone mosaic (window no. 1 from Hirsch and Miller, 1987) b) Test image.
Figure 2. The lowpass-filtered input image.

Figure 3. Sampling by averaging over circular disks.
Figure 4. Disordered samples of the input image. The samples correspond to the cone centers of the mosaic in Figure 1.

Figure 5. Disordered samples of the input image mapped onto an ordered hexagonal lattice.
Figure 6. Two-dimensional hexagonal tessellation of the foveal cone lattice.

Figure 7. The $\tau$ - transform maps each triangle defined by the tessellation to an equilateral triangle. The interior points of such a triangle are assigned to the interior points of the corresponding equilateral triangle using a trilinear interpolation function.
Figure 8. WKS reconstruction in the ordered hexagonal space, with appropriate interpolation between the ordered samples defined by Eq.(5).

Figure 9. The exterior hexagon is defined by the vectors $\vec{c} = (0, 2\sqrt{3}/3)$, $\vec{d} = (1, \sqrt{3}/3)$. These vectors generate the lattice that is the Fourier transform of the lattice generated by vectors $\vec{a}$ and $\vec{b}$. The interior hexagon is the lowpass kernel. If we sample a function by the lattice generated by $\vec{a}$ and $\vec{b}$, then aliasing will not occur if its spectrum belongs within the area of the lowpass kernel.
Figure 10. This image has been generated by direct application of the spatial mapping $\tau$ on the input image. It serves as a control for the WKS reconstruction in the ordered hexagonal space (Figure 8).

Figure 11. The final reconstructed image according to the proposed method. The samples correspond to the lattice points of Figure 1.
Figure 12. Step 1 of tessellation algorithm: Selection of the first set of points that will be assigned to a line in $\mathcal{X}$ space.

Figure 13. Step 2 of tessellation algorithm: Selection of the next set of points that will be assigned to the next line in $\mathcal{X}$ space. Each point is selected in such a way as to preserve the two-dimensional hexagonal topography.
Figure 14. The discrete Fourier transform of image samples on the disordered cone lattice (Figure 4). A CPI (cone positions effectively ignored) image reconstruction can be obtained by circular lowpass filtering in the frequency domain, followed by an inverse Fourier transform.

Figure 15. Result of CPI image reconstruction.
Figure 16. Result of "control" reconstruction.

Figure 17. The tessellated foveal cone lattice superimposed on the image reconstructed by the CPK method. This picture shows the dependence of reconstructed image quality on the spatial mapping $\tau$. 
Figure 18. "Functional" tessellation of the first two foveal windows. Squares indicate 'seeds' of disorder and circumscribe ordered subregions.
Figure 19. Two foveal subregions, defined according to a "functional" approach to tessellation, superimposed on the input image.
Figure 20. a) Result of "local" image reconstruction for the two regions appearing in Figure 19. b) Input image with regions in (a) shown for comparison.
Figure 21. A possible new "functional" windowing scheme for the first three foveal windows. Windows are defined with respect to two-dimensional hexagonal uniformity.